

On the non-attractive character of gravity in $f(R)$ theories

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(Dated: December 20, 2012)

In General Relativity without a cosmological constant a non-positive contribution from the space-time geometry to Raychaudhuri equation is found provided that particular energy conditions are assumed and regardless the considered solution of the Einstein's equations. This fact is usually interpreted as a manifestation of the attractive character of gravity. Nevertheless, a positive contribution to Raychaudhuri equation from space-time geometry should occur since this is the case in an accelerated expanding Robertson-Walker model for congruences followed by fundamental observers. Modified gravity theories provide the possibility of a positive contribution although the standard energy conditions are assumed. We address this important issue in the context of $f(R)$ theories, deriving explicit upper bounds for the contribution of space-time geometry to the Raychaudhuri equation. Then, we examine the parameter constraints for some paradigmatic $f(R)$ models in order to ensure a positive contribution to this equation.

PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k

I. INTRODUCTION

Since observational evidence of the accelerated expansion of the Universe was discovered [1], the cosmological evolution as predicted by General Relativity (GR) has been set in doubt. The reason is that a stress-energy tensor possessing strange features needs to be included in the Einstein's equations in order to account for this cosmic acceleration. This exotic cosmological fluid is usually referred to as dark energy (DE). In its simple form it is given by a cosmological constant with equation of state $p_\Lambda = -\rho_\Lambda$. However, instead of filling the Universe with exotic fluids, a reasonable hypothesis consists in modifying the cosmological field equations assuming alternative geometrical theories to GR. This approach has received the name of modified gravity theories and has drawn some attention in the last years [2].

Some examples are Lovelock theories, whose field equations are second-order differential equations in the metric [3]; Gauss-bonnet theories inspired in string theory that include a Gauss-Bonnet term in the Lagrangian [4]; scalar-tensor theories [5] or vector-tensor theories [6], in which gravitational interaction is not only mediated by the standard spin-2 graviton but also by scalar or vector modes respectively; metric theories derived by extra dimensional theories [7]; supergravity models [8], disformal theories [9], Lorentz violating and CPT breaking models of gravity [10]; or the so-called $f(R)$ theories, in which our work will be focused. $f(R)$ theories consist in modifying the Einstein-Hilbert Lagrangian by adding an arbitrary function of the Ricci scalar R (for recent reviews see [11]). From this approach, the equations derived from the new action are to be expected as a refinement of the standard Einstein's equations able to reproduce the correct predictions of GR while explaining the cosmic ac-

celeration. These theories may have strong effects on small scales, but if some restrictions are imposed, they are able to reproduce the cosmological history while being compatible with local gravity tests [12]. It is worth mentioning that the Einstein's equations with cosmological constant Λ are a particular case of these theories with $f(R) = -2\Lambda$.

In fact, the problem with the accelerated expansion of the Universe follows immediately from the consideration of the Friedmann's equations obtained assuming a Robertson-Walker (RW) cosmological model and a perfect fluid moving along the geodesic congruence followed by the fundamental observers. It is well-known that stress-energy tensors corresponding to standard fluids cannot be responsible for the accelerated expansion. From a more general point of view, the Friedmann's equation involving the acceleration of the scale factor results from the Raychaudhuri's equation assuming GR. This equation provides the expansion rate of a congruence of timelike or null geodesics (see [13–17] and recent review [18]). The Raychaudhuri equation plays an important role in the demonstration of the singularities theorems proved by Hawking and Penrose [13]. It is usually interpreted that the contribution of space-time geometry to this equation represents the attractive (or non-attractive) character of gravity. An analysis of this contribution [19] showed its geometrical interpretation as the mean curvature [20] in the direction of the congruence. Besides, it can easily be verified that for a RW cosmological model with a negative deceleration parameter this contribution is positive, i.e. the mean curvature in the direction of the fundamental congruence turns out to be positive [19]. Hence, the attractive character of gravity vanishes. From this analysis, it is clear that the accelerated expansion of the Universe may be in conflict with the attractive character of gravity.

In GR, the attractive character of gravity is assured by assuming the usual energy conditions [13, 14]. Therefore, a positive contribution to the Raychaudhuri equation from space-time geometry is not attainable in GR provided that these energy conditions hold. Nevertheless, this does not need to be the case in the context of modified gravity theories. In these theories, even if the usual energy conditions are assumed, a positive contribution to the Raychaudhuri equation from space-time geometry may be obtained. Moreover, an upper bound to the contribution of space-time geometry can be provided both in terms of the gravitational model and the metric under consideration. Using this upper bound and assuming the usual energy conditions, throughout this investigation we shall derive restrictions to $f(R)$ models in order to constrain their cosmological viability.

Energy conditions have been widely studied in the literature for different modified gravity theories. The authors of [21] generalized energy conditions for a perfect fluid in $f(R)$ theories by analogy with GR. In [22], the extended energy conditions of $f(R)$ are used to derive energy conditions in Brans-Dicke theories with a vanishing kinetic term in the Lagrangian using the equivalence between both theories. The energy conditions have also been studied for $f(R)$ theories with a non-minimal coupling to matter [23]. In [24], the same procedure is applied to Gauss-Bonnet theories. In [25] and [26], the authors considered Gauss-Bonnet theories with non-minimal coupling to matter and derive the corresponding energy conditions. All the aforementioned references followed the formalism first developed in [21]. This generalization of the energy conditions, as the authors of [21] themselves first acknowledged remains doubtful since there is no natural motivation but only an analogy with GR. This extension is in fact only motivated when the new terms appearing in the field equations are identified with physical fields. Nevertheless, these new terms may be understood as possessing only a geometrical meaning. Thus, there is no reason to assume any energy conditions on these terms.

Moreover, if these new energy conditions are assumed, the mean curvature in every timelike direction is non-positive by construction and, as already mentioned, for a RW space-time experiencing an accelerated expansion implies necessarily a positive mean curvature in the direction of the fundamental congruence. In particular, the stress-energy tensor associated with a cosmological constant Λ does not satisfy the usual energy conditions. This fact shows by itself the limitations of previous investigations assumptions in the most trivial modified gravity Lagrangian beyond pure GR.

This paper is organized as follows: First, in Section II, we summarize the standard energy conditions as considered in GR and then the role of the Raychaudhuri equation in the theorems of singularities is briefly discussed. Also, the procedure to be developed in the following section is also sketched. Section III is devoted to introduce the field equations for $f(R)$ theories in the metric for-

malism as well as the commonly assumed conditions for $f(R)$ models to be cosmologically viable. Then, in Section IV, we assume the energy conditions in the framework of $f(R)$ theories and present the inequalities that are obtained. We shall proceed by studying configurations of constant scalar curvature in Section V. These configurations will enable us to impose some constraints on the parameters of several relevant $f(R)$ models in order to get a positive contribution to the Raychaudhuri equation. Finally, we conclude our analysis by presenting our conclusions.

Throughout this study, we use a metric signature $(-, +, +, +)$ and our definition of the Riemann tensor is:

$$R_{abc}{}^d \equiv \partial_b \Gamma_{ac}^d - \partial_a \Gamma_{bc}^d + \Gamma_{ac}^e \Gamma_{eb}^d - \Gamma_{bc}^e \Gamma_{ea}^d, \quad (1)$$

$R_{ac} \equiv R_{abc}{}^b$ holds for the Ricci tensor and $R = R_a{}^a$ is the Ricci scalar. With this convention the usual Einstein's equations yield:

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab}. \quad (2)$$

From now on, we shall adopt $c = G = 1$. Furthermore, the stress-energy tensor corresponding to a perfect fluid is:

$$T_{ab} = \rho \xi_a \xi_b + p (g_{ab} + \xi_a \xi_b). \quad (3)$$

II. ENERGY CONDITIONS IN GENERAL RELATIVITY

The Raychaudhuri equation for timelike geodesics can be expressed as [14, 15]

$$\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - R_{ab} \xi^a \xi^b, \quad (4)$$

where θ , σ_{ab} and ω_{ab} are respectively the expansion, shear and twist of the congruence of timelike geodesics generated by the tangent vector field ξ^a and τ is an affine parameter. On the other hand, the analogous equation for null geodesics becomes [14, 16]

$$\frac{d\hat{\theta}}{d\lambda} = -\frac{1}{2} \hat{\theta}^2 - \hat{\sigma}_{ab} \hat{\sigma}^{ab} + \hat{\omega}_{ab} \hat{\omega}^{ab} - R_{ab} k^a k^b, \quad (5)$$

where k^a is the tangent vector field to a congruence of null geodesics and λ is an affine parameter. Let us recall that (4) and (5) are geometrical identities, thus they hold independently of the gravitational theory assumed.

In this investigation we are interested in the contribution of space-time geometry to the previous equations, i.e., $-R_{ab} \xi^a \xi^b$ and $-R_{ab} k^a k^b$. If a particular form for the metric tensor is assumed a priori, the Ricci tensor can be determined and then those contributions can be directly studied as it is the case for a RW metric [19] without considering any underlying gravitational theory. However, the metric tensor is in general unknown from the beginning, thus an expression for the Ricci tensor is

not at our disposal. In the latter scenario, the problem can be nonetheless tackled by using the field equations to obtain information about $-R_{ab}\xi^a\xi^b$ and $-R_{ab}k^ak^b$. This procedure leads to considering conditions to be imposed on the stress-energy tensor T_{ab} , the so-called energy conditions. Let us thus revise these conditions and some of their implications in GR both for timelike and null vectors.

Timelike vectors

The energy density of matter as measured by an observer with velocity ξ^a , is $T_{ab}\xi^a\xi^b$. It is reasonable that this density would be non-negative. This requirement is known as the weak energy condition (WEC)

$$T_{ab}\xi^a\xi^b \geq 0 \quad \text{WEC.} \quad (6)$$

Moreover, the dominant energy condition (DEC) ensures that the speed of the flux of energy is less than the speed of light, yielding

$$T_{ab}\xi^a T^{bc}\xi_c \leq 0 \quad \text{DEC,} \quad (7)$$

which expresses that the flux of energy, i.e. $-T^{ab}\xi_a$, is a timelike vector where the minus sign appears because we have chosen signature $(-, +, +, +)$. Furthermore, we are mainly interested in the expression $R_{ab}\xi^a\xi^b$. Using the usual Einstein's equations (2) and the subsequent relation between the Ricci scalar and the trace of the stress-energy tensor, i.e. $R = -8\pi T$, we obtain

$$\begin{aligned} R_{ab}\xi^a\xi^b &= 8\pi \left(T_{ab} - \frac{1}{2}Tg_{ab} \right) \xi^a\xi^b \\ &= 8\pi \left(T_{ab}\xi^a\xi^b + \frac{1}{2}T \right). \end{aligned} \quad (8)$$

It is customary to assume the positive sign of the r.h.s. of this equation since a distribution of standard matter would not result in a stress-energy tensor with pressure so large and negative as to make this member negative. This statement can be understood after replacing expression (3) in (8). Hence, stress-energy tensors for standard matter fluids satisfy the so-called strong energy condition (SEC)

$$T_{ab}\xi^a\xi^b \geq -\frac{1}{2}T, \quad \text{SEC.} \quad (9)$$

It is known that both dust matter and radiation satisfy the SEC. For a discussion about cases where this condition does not hold see [13]. In particular, a stress-energy tensor corresponding to a cosmological constant Λ fluid does not fulfill the SEC. We will discuss this case at the end of the section to avoid losing continuity in the discussion. Therefore, the SEC requires

$$R_{ab}\xi^a\xi^b \geq 0, \quad (10)$$

which may be interpreted, because of asserting a non-positive contribution to Raychaudhuri equation, as a

manifestation of the attractive character of gravity. It follows that the mean curvature [19, 20] in every timelike direction defined by

$$\mathcal{M}_{\xi^a} \equiv -R_{ab}\xi^a\xi^b \quad (11)$$

is negative or zero in GR provided that the SEC is assumed.

The usefulness of the Raychaudhuri equation in the singularity theorems is based upon the following result: if one chooses a congruence of timelike geodesics whose tangent vector field is locally hypersurface-orthogonal, then $\omega_{ab} = 0$ for all the congruence (as a consequence of Frobenius' theorem [14]) is obtained. The term $\sigma_{ab}\sigma^{ab}$ is non-negative and whenever $R_{ab}\xi^a\xi^b \geq 0$ is assumed, then

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0, \quad (12)$$

which implies

$$\theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{3}\tau. \quad (13)$$

This inequality tells us that a congruence initially converging ($\theta_0 \leq 0$) will converge until zero size in a finite time $\tau \leq 3/|\theta_0|$, or in a reversed sense, if the congruence is initially diverging $\theta_0 \geq 0$ it was focused until zero size in the past. This result is important in proving the theorems concerning singularities of Hawking and Penrose [13, 14]. Very often it is claimed that these theorems require energy conditions to hold, since for instance the SEC as we have just seen implies $R_{ab}\xi^a\xi^b \geq 0$. However, these theorems are essentially mathematical theorems independent of the gravitational theory. Energy conditions are necessary for these theorems to hold if and only if GR is assumed. Otherwise, the requirement would be $R_{ab}\xi^a\xi^b \geq 0$ for every non-spacelike vector. In Section IV, we shall assume the usual energy conditions in the framework of $f(R)$ theories but the sign of $R_{ab}\xi^a\xi^b$ will remain in principle undetermined.

Null vectors

Let us now consider a congruence of null geodesics. Just by replacing $\xi^a \rightarrow k^a$ in equation (8), one gets

$$R_{ab}k^ak^b = 8\pi \left(T_{ab} - \frac{1}{2}Tg_{ab} \right) k^ak^b = 8\pi T_{ab}k^ak^b. \quad (14)$$

Hence, the so-called null energy condition (NEC)

$$T_{ab}k^ak^b \geq 0 \quad \text{NEC,} \quad (15)$$

implies that $R_{ab}k^ak^b$ will be non-negative. NEC is fulfilled by continuity if the SEC is assumed, but it is also fulfilled by imposing the WEC. Then, the assumptions for a congruence of null geodesics to focus are weaker than those for a congruence of timelike geodesics sketched above. Reasoning in the same way as before, if a congruence of null geodesics is initially converging $\hat{\theta}_0 < 0$ it will

converge until zero size in a finite time $\tau \leq 2/|\hat{\theta}_0|$ or in the reversed sense.

The convergence of timelike and null geodesics in a finite time – in the future or in the past – is ensured in GR under the assumptions of the SEC and the NEC respectively. This result is usually known as *the geodesic focusing theorem* [13, 14].

Beyond General Relativity

If one considers a congruence whose tangent vector field is locally hypersurface-orthogonal, this means that, on the one hand $\omega_{ab} = 0$ and, on the other hand, the r.h.s. of the Raychaudhuri equation for timelike geodesics (4), has only non-positive contributions of the parameters of the congruence ($-\theta^2/3$ and $-\sigma_{ab}\sigma^{ab}$). If the observed acceleration of the Universe needs to be explained, a positive contribution of the space-time given by $\mathcal{M}_{\xi^a} > 0$ is required at least for some directions ξ^a . This is unfeasible in GR if the SEC is satisfied as can be seen from (10). This opens two ways of circumventing the unavoidable attractive character of gravity in GR: either to suppose that the SEC is not satisfied or to modify the Einstein's equations. An analogous discussion may be done for null geodesics replacing the SEC by the NEC.

In this work we shall consider the second scenario: the SEC will be located in a privileged place with respect to the Einstein's equations. We are thus assuming that standard matter satisfies the SEC and that the possibility of a positive contribution to Raychaudhuri equation through the mean curvature \mathcal{M}_{ξ^a} must be obtained from the $f(R)$ modified field equations. By proceeding in this way, an inequality involving $R_{ab}\xi^a\xi^b$ and terms depending on the gravitational theory under consideration will provide us an upper bound to the contribution of space-time geometry \mathcal{M}_{ξ^a} . This bound will allow us to derive some restrictions on the $f(R)$ models in order to get \mathcal{M}_{ξ^a} positive, or equivalently $R_{ab}\xi^a\xi^b$ negative.

As it was mentioned above, the cosmological constant stress-energy tensor does not satisfy the SEC and should not be regarded as an stress-energy term but as a particular $f(R) = -2\Lambda$ model.

III. $f(R)$ THEORIES

Let us consider the total action

$$S = S_{grav} + S_{matter}, \quad (16)$$

i.e. a gravitational action plus a matter action term that includes all matter fields. The modification of $f(R)$ theories to GR consists in assuming S_{grav} of the form

$$S_{grav} = \frac{1}{16\pi} \int (R + f(R)) \sqrt{|g|} d^4x, \quad (17)$$

where $f(R)$ is an arbitrary function of R and g is the determinant of the metric. The variation of (16) with respect to the metric tensor and using (17) and (19) yields

$$R_{ab}(1 + f'(R)) - \frac{1}{2}g_{ab}(R + f(R)) - (\nabla_a \nabla_b - g_{ab}\square) f'(R) = 8\pi T_{ab}. \quad (18)$$

where stress-energy tensor is defined as

$$T^{ab} \equiv -\frac{2}{\sqrt{|g|}} \frac{\delta S_{matter}}{\delta g_{ab}}. \quad (19)$$

Equations (18) are obtained in the metric formalism, i.e. the connection is assumed to be Levi-Civita connection. $f(R)$ theories have been proved to be able to reproduce the cosmological history from inflation to the actual accelerated expansion era. For instance, it has been showed that the current evolution of the Universe can be reproduced with certain $f(R)$ functions [27].

Let us remark here that equations (18) are fourth order differential equations. This is the reason of some strong instabilities that arise for certain $f(R)$ models, such as the Dolgov-Kawasaki instability for the model $f(R) = -\mu^4/R$ [28]. Moreover, there is a general instability known as Ostrograski instability associated with Lagrangian that contains non-linear second derivatives terms. However, it has been proved that one can avoid Ostrograski instabilities in $f(R)$ theories (cf. [29]). The Cauchy problem in $f(R)$ theories has also been considered [30] where the authors concluded that the problem is well-posed in the metric formalism. Further details as well as local and cosmological tests for $f(R)$ theories can be seen in [11, 12] among others.

Viability conditions of $f(R)$ theories

Some constraints are usually imposed on the $f(R)$ functions in order to provide consistent theories of gravity. We would like to highlight two clear results that we will take into account in our study:

1. $1 + f'(R) > 0$. This condition is imposed in order to ensure a positive effective gravitational constant $G_{eff} \equiv G/(1 + f'(R))$. It means that the main part of the contribution to the Einstein's equations conserves the sign [31]. This condition also guarantees the non-tachyonic character of the standard graviton.
2. $f''(R) \geq 0$. It ensures a stable gravitational stage. It is directly related to the presence of a positive mass in a high curvature regime for the scalar mode associated with this type of theories [32].

We will assumed both conditions for our discussion.

IV. ENERGY CONDITIONS IN $f(R)$ THEORIES

In this section we are interested in analogous equations to (8) and (14) when extended to $f(R)$ theories. Thus, by imposing the usual energy conditions, inequalities involving the terms $R_{ab}\xi^a\xi^b$ (or $R_{ab}k^ak^b$) and $f(R)$ extra geometrical terms will be obtained. These inequalities will provide us an upper bound for the contribution of space-time geometry to Raychaudhuri equation for time-like geodesics (4) and for null geodesics (5).

The usual approach in literature consisted of defining an effective stress-energy tensor by analogy with GR that includes the new geometrical terms in order to obtain an expression for $R_{ab}\xi^a\xi^b$. Then, analogous energy conditions on this effective tensor were imposed. By this procedure, a negative or zero contribution to Raychaudhuri equation from the term $\mathcal{M}_{\xi^a} = -R_{ab}\xi^a\xi^b$ for every time-like direction ξ^a is obtained whenever these analogous energy conditions hold. Nevertheless, as we have already mentioned, $\mathcal{M}_{\xi^a} > 0$ is satisfied for almost all timelike directions in a RW cosmological model with the present value of the deceleration parameter q_0 [19]. Therefore, if those extended energy conditions hold, the present accelerated expansion of the Universe cannot be accommodated.

Inequalities derivation

Let us first take the trace of equation (18) that can be recast as

$$R = \frac{-8\pi T - 2f(R) + 3\Box f'(R)}{1 - f'(R)}. \quad (20)$$

Therefore, from (18) together with (20) one gets

$$R_{ab}(1 + f'(R)) - \frac{1}{2}g_{ab}(Rf'(R) - f(R) + 3\Box f'(R)) - (\nabla_a\nabla_b - g_{ab}\Box)f'(R) = 8\pi\left(T_{ab} - \frac{1}{2}Tg_{ab}\right). \quad (21)$$

Contracting the last equation with $\xi^a\xi^b$, where ξ^a is a normalized timelike vector, $\xi^a\xi_a = -1$, we get

$$R_{ab}\xi^a\xi^b(1 + f'(R)) - \left(\xi^a\xi^b\nabla_a\nabla_b - \frac{1}{2}\Box\right)f'(R) + \frac{1}{2}(Rf'(R) - f(R)) = 8\pi\left(T_{ab}\xi^a\xi^b + \frac{1}{2}T\right). \quad (22)$$

If we consider a null vector k^a instead of a timelike vector, then multiplying (21) by k^ak^b the result becomes

$$R_{ab}k^ak^b(1 + f'(R)) - k^ak^b\nabla_a\nabla_b f'(R) = 8\pi T_{ab}k^ak^b. \quad (23)$$

At this stage, let us impose the SEC and the NEC to the standard cosmological fluids in the expressions (22)

and (23) respectively. After some manipulations, they become

$$R_{ab}\xi^a\xi^b \geq \frac{1}{2(1 + f'(R))} \times [f(R) - Rf'(R) + (2\xi^a\xi^b\nabla_a\nabla_b - \Box)f'(R)] \quad (24)$$

and

$$R_{ab}k^ak^b \geq \frac{1}{1 + f'(R)}k^ak^b\nabla_a\nabla_b f'(R), \quad (25)$$

where in both expressions $1 + f'(R)$ was assumed to be positive in order to guarantee $G_{eff} \equiv G/(1 + f'(R)) > 0$. Let us remind that our conclusions will be based upon this requirement. As a result of (24) and (25), one can conclude that although the SEC (NEC) have been assumed, in $f(R)$ theories the sign of $R_{ab}\xi^a\xi^b$ ($R_{ab}k^ak^b$) cannot be determined a priori. Thus, if a certain model $f(R)$ renders negative the right-hand side (r.h.s.) of (24) or (25), some freedom remains for $R_{ab}\xi^a\xi^b$ or $R_{ab}k^ak^b$ to be negative which may be interpreted as a repulsive force. Reminding now the definition (11), the inequalities (24) and (25) can be cast in the following way

$$\mathcal{M}_{\xi^a} = -R_{ab}\xi^a\xi^b \leq \frac{-1}{2(1 + f'(R))} \times [f(R) - Rf'(R) + (2\xi^a\xi^b\nabla_a\nabla_b - \Box)f'(R)], \quad (26)$$

$$-R_{ab}k^ak^b \leq \frac{-1}{1 + f'(R)}k^ak^b\nabla_a\nabla_b f'(R); \quad (27)$$

that provide upper bounds to the contribution of space-time geometry to the Raychaudhuri equation for timelike and null geodesics respectively.

Let us stress that in vacuum the inequalities (24) and (25) – or equivalently (26) and (27) – get saturated since the SEC/NEC energy conditions are trivially saturated. Consequently, if a $f(R)$ model renders the r.h.s. of (26) and (27) positive in vacuum scenario, then a positive contribution to the Raychaudhuri equation is automatically obtained for timelike geodesics and null geodesics respectively.

Let us focus on a short example. The Einstein's equations with a cosmological constant Λ become

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab}. \quad (28)$$

As we have previously noted, GR with a cosmological constant is equivalent to take $f(R) = -2\Lambda$. In this trivial case, the inequalities (26) and (27) become

$$\mathcal{M}_{\xi^a} \leq \Lambda, \quad -R_{ab}k^ak^b \leq 0. \quad (29)$$

The first inequality tells us that in the case of timelike geodesics a positive contribution to the Raychaudhuri equation from space-time geometry $\mathcal{M}_{\xi^a} > 0$ is possible provided that $\Lambda > 0$, which corresponds to the correct sign of Λ to provide cosmic acceleration.

Hence, it may be thought that a simple criterion to decide when a $f(R)$ model is able to render accelerated

expansion of the Universe has been obtained. Nonetheless, expressions (24) and (25) need to be evaluated at the solutions for (18) so the problem remains cumbersome. However, such a problem is absent when space-time configurations with constant scalar curvature R_0 are considered.

Constant scalar curvature solutions

Space-times both in vacuum and GR cosmological scenarios when studied at late times, with both radiation and dust being negligible with regard to a cosmological constant, are maximally symmetric, i.e. they possess a constant Gaussian curvature K_0 . This implies a constant scalar curvature $R_0 = 12K_0$ (but the reverse is not generally true). For this reason, it may be expected that solutions of constant scalar curvatures will be recovered at late times by physically viable $f(R)$ models.

Since the covariant derivatives of $f'(R)$ in solutions of constant scalar curvature ($R = R_0$) are zero, expressions (24) and (25) result respectively in

$$R_{ab}\xi^a\xi^b \geq \frac{f(R_0) - R_0f'(R_0)}{2(1 + f'(R_0))} \quad (30)$$

and

$$R_{ab}k^ak^b \geq 0. \quad (31)$$

A remarkable result follows from the inequality (31): In $f(R)$ theories, the condition for the null geodesic focusing theorem to hold, namely $R_{ab}k^ak^b \geq 0$, is satisfied in space-times of constant scalar curvature provided that the NEC is assumed as given by (15). It is worth noticing that this result does not depend upon the sign of R_0 nor upon the $f(R)$ model under consideration. Moreover, the NEC is only assumed in the standard stress-energy tensor for matter, not in the *effective* one usually defined after gathering all the new terms of the modified Einstein equations. Since the holographic principle [33] makes use of the null geodesic focusing theorem in order to ensure that light-sheets will eventually end, the previous result is of extraordinary importance when studying this principle in $f(R)$ theories.

In the rest of this investigation we shall focus on time-like geodesics. Therefore, the r.h.s. of (30) must be negative in order to allow $R_{ab}\xi^a\xi^b < 0$ or equivalently $\mathcal{M}_{\xi^a} > 0$. Thus, \mathcal{M}_{ξ^a} be bounded from above. Hence we impose

$$\frac{f(R_0) - R_0f'(R_0)}{2(1 + f'(R_0))} < 0, \quad (32)$$

and provided that $G_{eff} > 0$, we get

$$f(R_0) - R_0f'(R_0) < 0. \quad (33)$$

If we now consider the equation (20) in vacuum ($T = 0$) for constant scalar curvature solutions, the value of R_0

satisfies

$$R_0 = \frac{-2f(R_0)}{1 - f'(R_0)}, \quad (34)$$

which is an algebraic equation relating R_0 with the parameters of the $f(R)$ model under study. Although in general this equation cannot be solved analytically, there exist some $f(R)$ models for which a closed solution depending upon the parameters of the model can be found. Using the equation (34) in (33) one gets

$$\frac{f(R_0)}{1 - f'(R_0)} < 0, \quad (35)$$

that together with (34), implies

$$R_0 > 0. \quad (36)$$

Hence, a positive contribution to the Raychaudhuri equation from the space-time geometry \mathcal{M}_{ξ^a} for every time-like direction is obtained provided that $R_0 > 0$. This condition will constrain the parameters of different $f(R)$ models as will be seen in the next section.

In fact, there exists another straightforward way of getting (35) as follows: The solutions of (18) in vacuum with constant scalar curvature imply

$$R_{ab} (1 + f'(R_0)) - \frac{1}{2} g_{ab} (R_0 + f(R_0)) = 0 \quad (37)$$

and consequently

$$R_{ab} = \frac{1}{2} \frac{R_0 + f(R_0)}{1 + f'(R_0)} g_{ab} = \frac{R_0}{4} g_{ab}, \quad (38)$$

where (34) has been used in the last equality. It means that the allowed $f(R)$ space-times with constant scalar curvature in vacuum are Einstein spaces [20]. Thus, if a negative value of $R_{ab}\xi^a\xi^b$ is required in order to have $\mathcal{M}_{\xi^a} > 0$ the condition $R_0 > 0$ needs to be accomplished. However, one must keep in mind that the inequality (33) is more general since it allows us to have $\mathcal{M}_{\xi^a} > 0$ even for non-vacuum scenarios.

If one consider a maximally symmetric space-time, the condition (36) implies that in order to get a positive contribution to the Raychadhuri equation the space-time must be de Sitter ($R_0 > 0$). The constraints on the parameters of different $f(R)$ models explored in the following guarantee that such a space-time exists as a solution of the modified Einstein's equations (18).

V. $f(R)$ MODELS

Let us now study some $f(R)$ models in vacuum to illustrate the previous results:

• Model I $f(R) = \alpha |R|^\beta$

This model encompasses a wide variety of proposals available in the literature. For instance, the

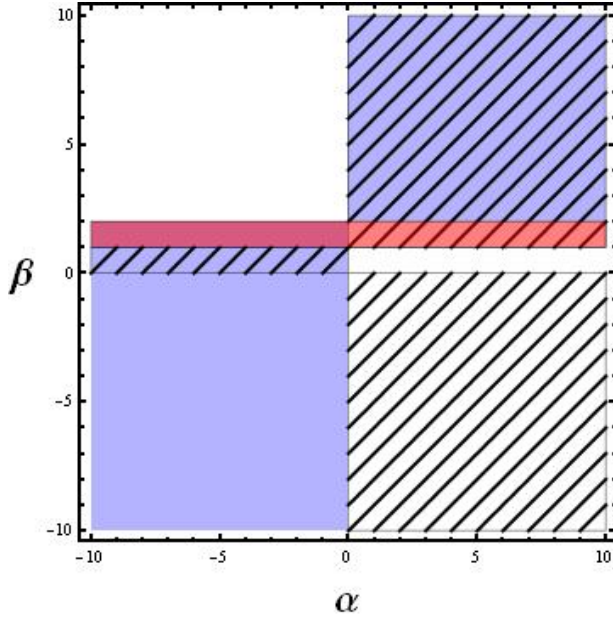


Figure 1. (α, β) plane for Model I in vacuum: The pairs (α, β) which lying in the blue region are those that satisfy $\alpha(\beta - 2) > 0$ and consequently $R_0 > 0$. The meshed zone fulfills $f''(R_0) \geq 0$ which is a stability condition commented in Section III. The red zone does not satisfy the condition $G_{eff} = G/(1 + f'(R_0)) > 0$ and therefore the inequality (35) is not valid there as we have already commented in the previous section. The parameters that provide a positive contribution from the space-time geometry, i.e. $\mathcal{M}_{\xi^a} = -R_{ab}\xi^a\xi^b > 0$, to Raychaudhuri equation for congruences of timelike geodesics are those of the blue zone excluding the red zone about which no statement can be done with our discussion.

case with $\beta = 2$ has been considered by Starobinsky in relation with the inflationary epoch in [34] and more recently in [35] as a candidate to take account of the dark matter. The case $\beta < 1$ and in particular $\beta = -1$ was proposed in [36] as a possible mechanism to provide cosmological acceleration, although it is currently ruled out. For this model, the curvature scalar can be obtained from (34) yielding

$$R_0 = \pm \left[\frac{\pm 1}{\alpha(\beta - 2)} \right]^{\frac{1}{\beta - 1}}, \quad (39)$$

where the sign depends upon the sign of R_0 because of the derivative of the absolute value (plus signs for $R_0 > 0$ and minus signs for $R_0 < 0$). There is also a trivial solution with $R_0 = 0$ which is of no interest for our discussion. Since we are interested in $R_0 > 0$ we take the expression with the plus signs. In order to a positive constant scalar curvature exist the parameters must obey

$$\alpha(\beta - 2) > 0. \quad (40)$$

Note that for $\beta = 2$ the only constant scalar curvature in vacuum is $R_0 = 0$.

Regions where the conditions $\alpha(\beta - 2) > 0$ and $f''(R_0) \geq 0$ hold are plotted in Figure 1. The region where $G_{eff} = G/(1 + f'(R_0)) > 0$ does not hold is also represented. Since this viability condition of $f(R)$ has been assumed in deriving the inequality (35), our discussion is not valid for the values of the parameters falling in that region. Let us stress that the conditions $f''(R_0) \geq 0$ and $G_{eff} > 0$ are evaluated in the corresponding value of the constant scalar curvature R_0 which depends on the parameters. It means that these conditions will be satisfied by the parameters that fall in the corresponding regions in the case of constant scalar curvature solutions, not for every solution of the equation (18). This consideration remains valid for all the models studied in this section. In Figure 1 one can also see that for $\beta > 0$ there are regions where both requirements, namely $R_0 > 0$ and $f''(R_0) \geq 0$, hold. For $\alpha < 0$, β must be restricted to the interval $(0, 1)$; on the contrary, for $\alpha > 0$, it must be $\beta > 2$; in order both requirements to hold.

• Model II $f(R) = R^\alpha \exp(\beta/R) - R$

This model was discussed for $\alpha = 1$ in [37] and more recently in [38]. By proceeding as for the previous model, one gets from equation (34)

$$R_0 = \frac{\beta}{\alpha - 2}. \quad (41)$$

Thus, in order to ensure a positive contribution to the Raychaudhuri equation from the space-time geometry, i.e. $\mathcal{M}_{\xi^a} = -R_{ab}\xi^a\xi^b > 0$ one must impose

$$\frac{\beta}{\alpha - 2} > 0. \quad (42)$$

Note that for $\alpha = 2$, the only constant scalar curvature vacuum solution is $R_0 = 0$. The same conditions as for the previous model are plotted in Figure 2. Let us remind that the conditions $G_{eff} > 0$ and $f''(R_0) \geq 0$ are evaluated in R_0 which depends on the parameters of the model.

Let us stress that for this case the value of $f''(R_0)$ depends on the sign of R_0 . Since we are interested in the region where $R_0 > 0$ holds, the condition $f''(R_0) \geq 0$ is plotted after assuming $R_0 > 0$. For this model, in the zone where $G_{eff} > 0$ holds, all the parameters that provide a positive scalar curvature $R_0 > 0$ also fulfill $f''(R_0) \geq 0$.

• Model III $f(R) = R [\log(\alpha R)]^\beta - R$

This model was also considered in [37, 38]. In this case, equation renders (34)

$$R_0 = \frac{1}{\alpha} \exp(\beta). \quad (43)$$

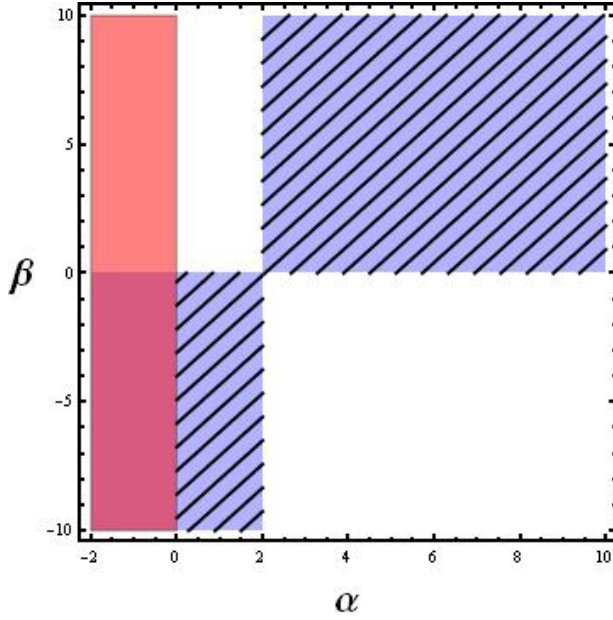


Figure 2. (α, β) plane for Model II in vacuum: For the parameters in the blue zone a positive constant scalar curvature exist. The condition $f''(R_0) \geq 0$ is fulfilled in the meshed zone. Let us remark that in this model the condition $f''(R_0) \geq 0$ depends on the sign of R_0 and the value of α . For simplicity, the condition $f''(R_0) \geq 0$ is plotted after assuming $R_0 > 0$ since it is the case in which we are interested. Moreover, if $R_0 > 0$ the condition $f''(R_0) \geq 0$ does not depend on the value of α . In the red region the condition $G_{eff} = G/(1 + f'(R_0)) > 0$ does not hold and consequently the inequality (35) does not apply there.

Therefore, for this model the condition guaranteeing both $R_0 > 0$ and a positive contribution to Raychaudhuri equation for timelike geodesics from the space-time geometry is $\alpha > 0$.

The same conditions considered for the previous models are plotted in Figure 3. For this case, there exists also a region where both conditions $R_0 > 0$ and $f''(R_0) \geq 0$ are satisfied, namely for $\alpha > 0$ and $\beta > 1/2$.

• **Model IV** $f(R) = -\gamma \frac{\kappa(\frac{R}{\gamma})^n}{1+\delta(\frac{R}{\gamma})^n}$

This model has been proposed in [39] as cosmological viable attracting much attention in the last years. In order to illustrate our procedure, let us consider the particular case with $n = 1$

$$f(R) = -\frac{\alpha R}{1 + \beta R}, \quad (44)$$

where a trivial redefinition of the parameters has been performed. The constant scalar curvature of this model in vacuum becomes

$$R_{\pm} = \frac{\alpha - 1}{\beta} \pm \frac{\sqrt{\alpha(\alpha - 1)}}{\beta}. \quad (45)$$

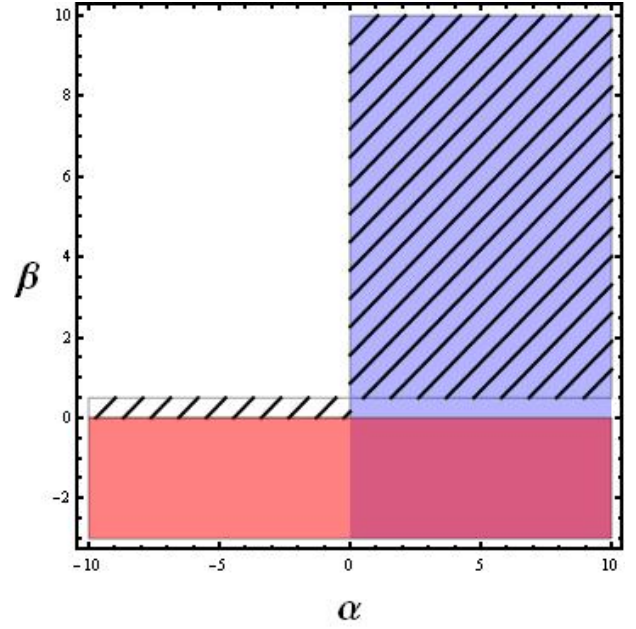


Figure 3. (α, β) plane for Model III in vacuum: As for the previous figures, $R_0 > 0$ is obtained with parameters in the blue zone. In the meshed zone $f''(R_0) \geq 0$. $G_{eff} = G/(1 + f'(R_0)) > 0$ is not fulfilled in the red zone and thus the inequality (35) is not valid there. Let us remark that the condition $f''(R_0) \geq 0$ depends on the sign of β . As we are interested in region where $G_{eff} > 0$ holds, it is plot the region where $f''(R_0) \geq 0$ assuming $\beta > 0$.

It follows that only for $\alpha(\alpha - 1) \geq 0$ constant scalar curvature solutions exist. Therefore, imposing the constraint $R_{\pm} > 0$ we get

$$\frac{\alpha - 1}{\beta} \pm \frac{\sqrt{\alpha(\alpha - 1)}}{\beta} > 0. \quad (46)$$

Analogous plots to the previous models are shown in Figure 4. For this model, two different figures are shown since there are two possible values of R_0 , namely R_{\pm} . For the case R_- , the regions $R_- > 0$ and $f''(R_-) \geq 0$ do not overlap. On the other hand, for the case R_+ , all the parameters pairs providing a positive constant scalar curvature $R_+ > 0$ also fulfill $f''(R_+) \geq 0$. Furthermore, for R_+ case, the condition G_{eff} is always satisfied.

VI. CONCLUSIONS

In this investigation we have studied the contribution of space-time geometry to the Raychaudhuri equation for timelike and null geodesics. In General Relativity without a cosmological constant, once the usual energy conditions are assumed it is not possible to obtain a positive contribution from the space-time geometry to the Raychaudhuri equation for timelike and null geodesics.

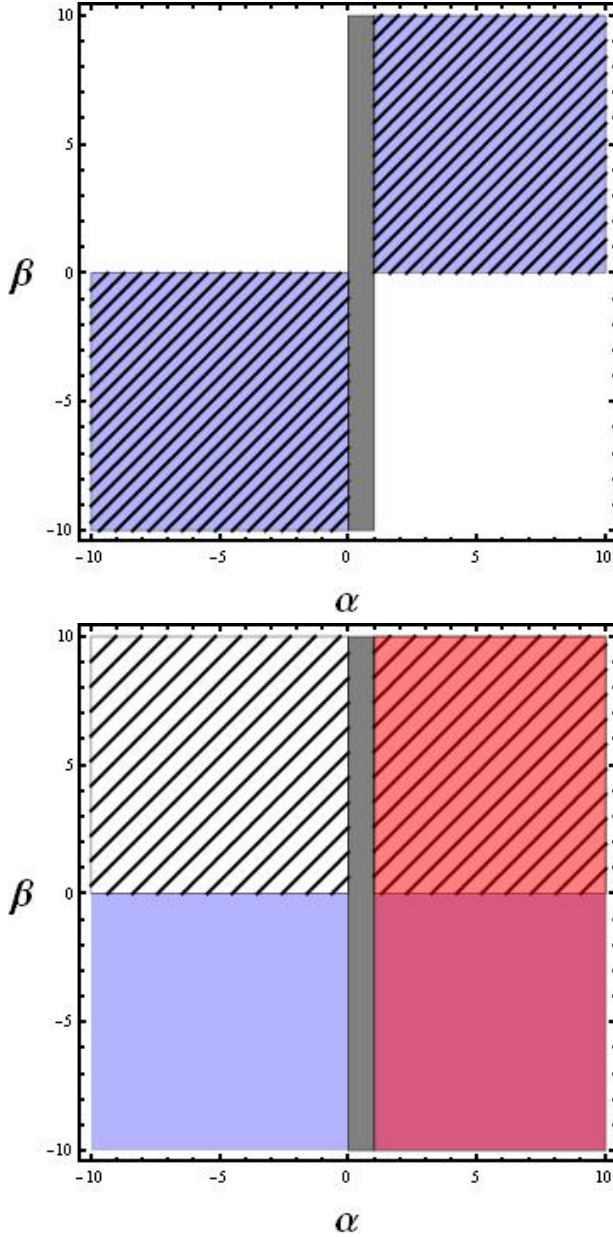


Figure 4. (α, β) plane for Model IV in vacuum: R_+ is considered at the upper panel whereas the lower panel considers R_- . The blue zone represents the region where $R_+ > 0$ ($R_- > 0$). In the meshed zone the condition $f''(R_+) \geq 0$ ($f''(R_-) \geq 0$) holds. The red zone parameters can not be considered in our discussion since $G_{eff} = G/(1 + f'(R_+)) > 0$ ($G_{eff} = G/(1 + f'(R_-)) > 0$) does not hold there. Finally, the parameters that fall in the grey zone do not fulfill $\alpha(\alpha - 1) \geq 0$ which is a necessary condition in this model in order to have a solution with constant scalar curvature.

Nonetheless, a positive contribution from space-time geometry to the Raychaudhuri equation is obtained for many timelike directions in the present Universe [19].

We have proved that in $f(R)$ modified gravity theories although the same energy conditions as in General

Relativity are assumed, the fact of getting a positive contribution to the Raychaudhuri equation from space-time is allowed.

We have derived two inequalities that bound from above this contribution for congruences of both timelike and null geodesics. In order to allow a positive contribution to the Raychaudhuri equation, these upper bounds must be positive. The limitation with the obtained inequalities is that in general in order to extract some information, a metric solution of the modified Einstein equations must be used. Nevertheless, in the cosmological relevant case of constant scalar curvature R_0 solutions, such a knowledge is not required. Under this assumption, it was obtained that $R_{ab}k^ak^b \geq 0$ where k^a is a null vector. This is the condition needed for the null geodesic focusing theorem to hold. Thus, this theorem remains valid in the context of $f(R)$ theories for space-times with constant scalar curvature, regardless the $f(R)$ model considered and the matter content (provided that the Null Energy Condition is fulfilled). This result acquires a remarkable importance when dealing with the holographic principle in the context of $f(R)$ theories. Since if the null focusing theorem holds for this scenario in fourth order gravity theories, the light-sheets will eventually end.

Finally, in vacuum scenarios for space-times of constant scalar curvature R_0 , we derived constraints on the parameters of paradigmatic $f(R)$ models guaranteeing a positive contribution to the Raychaudhuri equation. We conclude that for the models under consideration, there exist parameters values that allow the desired contribution while satisfying the condition $f''(R_0) \geq 0$, which ensures the stability of the solutions.

In this investigation, usual energy conditions were solely imposed upon the cosmological standard fluids whereas the extra $f(R)$ terms were considered as geometrical terms. This approach renders a more realistic analysis of the viability of $f(R)$ models and has been studied here for the first time. Furthermore, the sketched procedure developed constitutes a straightforward and systematic approach to decide whether a particular $f(R)$ model could generate cosmological acceleration. For this purpose, we have paid particular attention to constant curvature solutions. With the tools presented in this investigation, analysis can be extended to more complicated cosmological scenarios and other alternative gravity theories beyond the Concordance model [40].

ACKNOWLEDGMENTS

This work has been supported by MICINN (Spain) projects numbers FIS2011-23000, FPA2011-27853-C02-01 and Consolider-Ingenio MULTIDARK CSD2009-00064. AdlCD acknowledges financial support from NRF and URC research fellowships (South Africa).

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